22/10/15

Άσκ. A15 σελ. 11

$$\frac{dy}{dx} = \frac{(1+y)^2}{\underbrace{x-x^2+xy}_{x(1+y)-x^2}} , \; y(1)=1. \Rightarrow \frac{dz}{dz} = \frac{z^2}{xz-x^2}$$

$z = 1+y$

$$y'(x) = \frac{f(x)}{g(x)}$$

$$\frac{dy}{dx} = \frac{P(x,y)}{Q(x,y)}$$

$\underbrace{M(x,y)\,dx + N(x,y)\,dy = 0}$, $\quad M, N \in C(I)$ (E)

$\exists f$: Ω τόπος του $\mathbb{R}^2$ με $df = 0$

$$df = \underbrace{\frac{\partial f}{\partial x}}_{M} dx + \underbrace{\frac{\partial f}{\partial y}}_{N} dy$$

τότε $f(x,y) = c$.

(E): άμεσα ολ/κη : $df = c$

$$(C): \frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

π.χ. Άσκηση 2iii σελ. 48

$$\underbrace{(1+y^2+xy^2)}_{M(x,y)}dx + \underbrace{(x^2y+y+2xy)}_{N(x,y)}dy = 0$$

$\frac{df}{dx}$

$$\frac{\partial M}{\partial y} = 2y + 2xy \qquad \frac{\partial N}{\partial x} = 2xy + 2y$$

$$f(x,y) = \int (1+y^2+xy^2)\,dx + h(y) \Rightarrow$$

$$\Rightarrow f(x,y) = x + xy^2 + \frac{x^2y^2}{2} + h(y)$$

Είναι $\frac{\partial f}{\partial y}(x,y) = 2xy + x^2 y + h'(y)$

Θα πρέπει να είναι $x^2 y + y + 2xy$ $\Rightarrow$

$\Rightarrow h'(y) = y \Rightarrow \boxed{h(y) = \frac{y^2}{2} + c}$

Επομένως $f(x,y) = x + xy^2 + \frac{x^2 y^2}{2} + \frac{y^2}{2} = C$ $\Rightarrow \boxed{C' = 3} \Leftrightarrow$

$y(1) = 1 \Rightarrow x = 1$, $y = 1$

$\Leftrightarrow \boxed{x + xy^2 + \frac{x^2 y^2}{2} + \frac{y^2}{2} = 3}$

Επαληθεύω αν $\underbrace{\frac{\partial f}{\partial x}}_{M} dx + \underbrace{\frac{\partial f}{\partial x}}_{N} dy = 0$

αν $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = P(x) \Rightarrow P(x) = e^{\int P(x)}$

Τότε: $P(x,y) M(x,y) + P(x,y) N(x,y) = 0$.

ή

$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = Q(y) \Rightarrow P(y) = e^{\int Q(y)}$

π.χ. $\underbrace{y}_{M} dx + \underbrace{2x}_{N} dy = 0$

$\frac{\partial M}{\partial y}(x,y) = 1 \neq \frac{\partial N}{\partial x}(x,y) = 2$

$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{1-2}{2x} = -\frac{1}{2x} = P(x)$

άρα ολ. παράγοντας: $e^{\int -\frac{1}{2x} dx} = e^{-\frac{1}{2}\log x} = -\frac{1}{\sqrt{x}}$

άρα $-\frac{y}{\sqrt{x}}dx + \frac{2}{\sqrt{x}}dy = 0 \to \dots \to \boxed{y(x) = \frac{1}{2\sqrt{x}}}$

αλλιώς λύνεται και/κ χωρ. μεταβλ.

π.χ. $\underbrace{(y+xy^2)}_{M}dx - \underbrace{x}_{N}dy = 0$ → Η μηδενική λύση είναι λύση της εξίσωσης.

$$\frac{\partial M}{\partial y} = 1+2xy \neq \frac{\partial N}{\partial x} = -1$$

$$\frac{\frac{\partial N}{\partial x} - \frac{\partial M}{y}}{M} = \frac{-1-1-2xy}{y+xy^2} = \frac{-2(1+xy)}{y(1+xy)} = \frac{-2}{y} = Q(y)$$

$$\to \rho(x,y) = e^{\int -\frac{2}{y}dy} = \frac{1}{y^2}$$

άρα $\left(\frac{1}{y}+x\right)dx - \frac{x}{y^2}dy = 0$

$$\frac{\partial}{\partial y} = -\frac{1}{y^2} \qquad \frac{\partial}{\partial x} = -\frac{1}{y}$$

$$f(x,y) = \int\left(\frac{1}{y}+x\right)dx + h(y) = \frac{x}{y} + \frac{x^2}{2} + h(y)$$

$$f(x,y) = \int\left(-\frac{x}{y^2}\right)dy + g(x) = \frac{x}{y} + g(x)$$

→ $f(x,y) = \frac{x}{y} + \frac{x^2}{2}$

$\frac{x}{y} + \frac{x^2}{2} = c$

Άσκηση: $(4x^{-4}y^2 - 2x^{-2}y)dx + (3x^{-3}y + x^{-1})dy = 0$

→ $x^n y^m$

$$\to (4x^{n-4}y^{m+2} - 2x^{n-2}y^{m+1})dx + (3x^{n-3}y^{m+1} + x^{n-1}y^m)dy = 0$$

$$\frac{\partial M}{\partial y} = 4(m+2)x^{n-4}y^{m+1} - 2x^{n-2}(m+1)y^{m}$$

$$\frac{\partial N}{\partial x} = 3(m-3)x^{m-4}y^{n+1} + (m-1)x^{m-2}y^{n}$$

$$\left.\begin{array}{l} 4(n+2) = 3(m-3) \\ 2(n+1) = m-1 \end{array}\right\| \Rightarrow \begin{array}{l} 4n-3m = -17 \\ 2n-m = -3 \end{array}$$

$\longmapsto$

**Ασκ**: Να λυθεί $(y^3 - x^2y)dx + (x^3 - xy^2)dy = 0 \Leftrightarrow$

$\Leftrightarrow y(y^2 - x^2)dx + x(x^2 - y^2)dy = 0 \Leftrightarrow$

$\Leftrightarrow (y^2 - x^2)(ydx - xdy) = 0$

$\downarrow$ $\qquad\qquad \downarrow$

$y = \pm x \qquad$ χωρ. μεταβλ.

$y = 0$.

$\longmapsto$

B' τάξης $\longrightarrow$ A' τάξης

(A) $y''(x) = f(x, y'(x))$ // θέτω $u = y' \Rightarrow u' = y'' \rightsquigarrow u' = f(x,u)$

Παραδ. ① $y'' + (y')^2 + y' = 0$ // $y(0) = 0,\ y'(0) = 1$.

$y' = z \Rightarrow$

$$\Rightarrow z' + z^2 + z = 0 \Rightarrow \frac{dz}{dx} = -(z^2+z) \Rightarrow \frac{dz}{z^2+z} = -dx \Rightarrow$$

$$\Rightarrow \int\left(\frac{1}{z} - \frac{1}{z+1}\right)dz = -\int dx + c \Rightarrow \log\frac{z}{z+1} = c - x \Rightarrow \frac{z}{z+1} = e^{c-x}$$

$\longmapsto$

$$\int \frac{1}{ce^x - 1}dx = \int \frac{e^{-x}}{c - e^{-x}}dx = \ln|c - e^{-x}|$$

$\longmapsto$

(B) $y'' = f(y, y')$ // $z = y' \Rightarrow y'' = \frac{dy'}{dx} = \frac{dy'}{dy}\cdot\frac{dy}{dx} = z\frac{dz}{dy}$

δηλ. $z\frac{dz}{dy} = f(y,z)$ (εξ. α' τάξης με ανεξ. το $y$ και άγνωστο το $z$)

Ασκ. 2 σελ. 52

$$yy'' - (y')^2 = y^2 y'$$
$$z = y'$$
$$y'' = \frac{dz}{dy} z$$
$$\Rightarrow \quad yz\frac{dz}{dy} - z^2 = y^2 z$$

$z=0$: λύση

$$\Rightarrow \quad y\frac{dz}{dy} - z = y^2 \Rightarrow$$

$$\Rightarrow \boxed{\frac{dz}{dy} - \frac{1}{y} z = y} \Rightarrow \quad z = c|y| + y^2$$

$$y' = cy + y^2$$